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I. Babuska
Institute for Physical Science and Technology
University of Maryland, College Park, MD 20742

and

J. E. Osborn
Department of Mathematics
University of Maryland
College Park, MD 20742

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by

I. Babuška* and J.E. Osborn**

* Institute for Physical Science and Technology and Department of Mathematics, University of Maryland, College Park, MD 20742. The work of this author was partially supported by the Office of National Research under Contract N00014-85-K-0169 and by the National Science Foundation under Grant DMS-85-16191.

** Department of Mathematics, University of Maryland, College Park, MD 20742. The work of this author was partially supported by the National Science Foundation under Grant DMS-84-10324. Part of the work was done while this author was supported as a visitor at the Institute for Mathematics and its Applications, University of Minnesota.

1. Introduction

In this paper ~~we establish~~ some refined estimates for the approximation of the eigenvalues and eigenvectors of selfadjoint eigenvalue problems by finite element or, more generally, Galerkin methods. Suppose λ is an eigenvalue of multiplicity q of a selfadjoint problem and let $M(\lambda)$ denote the space of eigenvectors corresponding to λ. Denote by $\|\cdot\|_B$ the energy norm for the problem. Let $\{S_h\}_{0<h}$ be the family of finite dimensional approximation spaces employed in the Galerkin method. λ will be approximated from above by q of the Galerkin approximate eigenvalues:

$$\lambda \le \lambda_{h,1} \le \cdots \le \lambda_{h,q} ,$$

$$\lambda \cong \lambda_{h,1} , \cdots , \lambda_{h,q} .$$

Let u, with $\|u\|_B = 1$, denote an eigenvector corresponding to λ, and let $u_{h,1}, \cdots, u_{h,q}$, with $\|u_{h,k}\|_B = 1$, denote the Galerkin eigenvectors corresponding to $\lambda_{h,1}, \cdots, \lambda_{h,q}$, respectively.

It is well-known that

$$\lambda_{h,k} - \lambda \le C \sup_{\substack{u \in M(\lambda) \\ \|u\|_B = 1}} \inf_{\chi \in S_h} \|u - \chi\|_B^2 , \quad k = 1, \cdots, q, \tag{1.1}$$

and that there is a $u_k = u_k(h) \in M(\lambda)$, with $\|u_k\|_B = 1$, such that

$$\|u_{h,k} - u_k\|_B \le C \sup_{\substack{u \in M(\lambda) \\ \|u\|_B = 1}} \inf_{\chi \in S_h} \|u - \chi\|_B , \quad k = 1, \cdots, q. \tag{1.2}$$

In [7,8] Chatelin proved the following refinements of (1.1) and (1.2):

$$\|u - E_h u\|_B = r_h^{(a)} \inf_{\chi \in S_h} \|u-\chi\|_B \quad \forall\, u \in M(\lambda), \tag{1.3a}$$

$$\|u_{h,k} - Eu_{h,k}\|_B = r_h^{(b)} \inf_{\chi \in S_h} \|Eu_{h,k}-\chi\|_B, \quad k = 1,\cdots,q, \tag{1.3b}$$

and

$$\|(\lambda_{h,k} - \lambda)/\lambda\|_B = r_h^{(c)} \inf_{\chi \in S_h} \|Eu_{h,k}-\chi\|_B^2, \quad k = 1,\cdots,q, \tag{1.3c}$$

where E denotes the orthogonal projection of the energy space onto $M(\lambda)$ and E_h the orthogonal projection onto $\mathrm{span}\{u_{h,1},\cdots,u_{h,q}\}$ and where $r_h^{(\ell)} \to 1$ as $h \to 0$, for $\ell = a,b,c$.

The purpose of this paper is twofold. The first purpose is to establish an estimate for $|r_h^{(\ell)}-1|$. We show that

$$|r_h^{(\ell)}-1| \le d\eta^2(h), \tag{1.4}$$

where $\eta(h)$ is a certain measure of the approximability property of $\{S_h\}$; for the definition of η see Section 3. This is done in Section 4.

In [3] the authors established the estimate

$$\lambda_{h,1} - \lambda = C \inf_{\substack{u \in M(\lambda) \\ \|u\|_B=1}} \inf_{\chi \in S_h} \|u-\chi\|_B^2 , \tag{1.5}$$

which is an improvement over (1.1) and (1.3c) in the case of a multiple eigenvalue. [3] also contains estimates for $\lambda_{h,k}-\lambda, k=2,\cdots,q$, and for $\|u_{h,k}-u\|_B$, $k = 1,\cdots,q$, which are improvements of (1.1) and (1.3c) and of (1.2) and (1.3a,b), respectively. The second purpose of the paper is to present a simplified proof and an extension of the results in [3]. This is done in Section 5.

In Section 2 we give a precise statement of class of eigenvalue problems and approximation methods we will consider. Section 3 contains some background information.

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2. Setting for the Problem

Suppose H is a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, respectively, and suppose we are given two symmetric bilinear forms $B(u,v)$ and $D(u,v)$ on $H \times H$. $B(u,v)$ is assumed to satisfy

(2.1) $$|B(u,v)| \le C_1\|u\|\,\|v\|,\ \forall\ u,v \in H$$

and

(2.2) $$C_0\|u\|^2 \le B(u,u),\ \forall u \in H,\ \text{with}\ C_2 > 0.$$

It follows from (2.1) and (2.2) that $(u,v)_B = B(u,v)$ and $\|u\|_B = B(u,u)^{1/2}$ are equivalent to (u,v) and $\|u\|$, respectively. Regarding D we assume

(2.3) $$0 < D(u,u),\ \forall\ 0 \ne u \in H$$

and that

(2.4) $$\|u\|_D = D(u,u)^{1/2}$$

is compact with respect to $\|\cdot\|$, i.e., from any sequence which is bounded in $\|\cdot\|$, one can extract a subsequence which is Cauchy in $\|\cdot\|_D$. For the remainder of this paper we will use $B(u,v)$ and $\|\cdot\|_B$ as the inner product and norm on H and denote this space by H_B.

We then consider the variationally formulated, selfadjoint eigenvalue problem,

$$
(2.5) \qquad \begin{cases} \text{Seek } \lambda \text{ (real) and } 0 \neq u \in H_B \text{ satisfying} \\ B(u,v) = \lambda D(u,v), \ \forall\, v \in H_B . \end{cases}
$$

Under the assumptions we have made, (2.5) has a sequence of eigenvalues

$$
0 < \lambda_1 \le \lambda_2 \le \cdots \nearrow +\infty
$$

and corresponding eigenvectors

$$
u_1, u_2, \cdots,
$$

which can be chosen to satisfy

$$
(2.6) \qquad B(u_i, u_j) = \lambda_i D(u_i, u_j) = \delta_{ij}, \ i,j=1,2,\cdots \ .
$$

The eigenvalues and eigenvectors satisfy the following well-known variational principles:

$$
(2.7) \qquad \lambda_k = \min_{\substack{u \in H_B \\ B(u,u_i)=0 \\ i=1,2,\cdots,k-1}} \frac{B(u,u)}{D(u,u)} = \frac{B(u_k,u_k)}{D(u_k,u_k)}, \ k = 1,2,\cdots
$$

(the minimum principle)

and

$$
(2.8) \qquad \lambda_k = \min_{\substack{V_k \subset H_B \\ \dim V_k = k}} \max_{u \in V_k} \frac{B(u,u)}{D(u,u)} = \max_{u \in U_k = \mathrm{sp}(u_1,\cdots,u_k)} \frac{B(u,u)}{D(u,u)}, \ k=1,2,\cdots
$$

(the minimum-maximum principle).

For any λ_k we let

$$
(2.9) \qquad M = M(\lambda_k) = \{u\colon u \text{ is an eigenvector of (2.5) corresponding to } \lambda_k\}.
$$

We shall be interested in approximating the eigenpairs of (2.5) by finite element or, more generally, Galerkin methods. Toward this end we

suppose we are given a (one parameter) family $\left\{S_h\right\}_{0<h\le 1}$ of finite dimensional subspaces $S_h \subset H_B$ and we consider the eigenvalue problem,

$$(2.10)\qquad \begin{cases} \text{Seek } \lambda_h(\text{real}),\ 0 \neq u_h \in S_h \quad \text{satisfying} \\ B(u_h, v) = \lambda_h D(u_h, v),\ \forall\, v \in S_h\ . \end{cases}$$

The eigenpairs (λ_h, u_h) of (2.10) are then viewed as approximations to the eigenpairs $(\lambda.u)$ of (2.5). (2.10) is called the Galerkin method determined by the subspaces $\{S_h\}$ for the approximation of the eigenvalues and eigenvectors of (2.5). We will also sometimes refer to problem (2.10) as the Galerkin approximation of problem (2.5). (2.10) has a sequence of eigenvalues

$$0 < \lambda_{h,1} \le \lambda_{h,2} \le \cdots \le \lambda_{h,N}\ ,\ N = \dim\ S_h,$$

and corresponding eigenvectors

$$u_{h,1}, u_{h,2}, \cdots, u_{h,N}\ ,$$

which can be chosen to satisfy

$$(2.11)\qquad B(u_{u,i}, u_{h,j}) = \lambda_{h,i} D(u_{h,i}, u_{h,j}) = \delta_{ij}, i,j=1,\cdots,N\ .$$

The $(\lambda_{h,j}, u_{h,j})$ are referred to as the approximate eigenpairs, while (λ_j, u_i) are referred to as the exact eigenpairs of (2.5). Maximum and minimum-maximum principles analogous to (2.7) and (2.8) hold for problem (2.10); they are obtained from (2.7) and (2.8) by replacing H_B by S_h and letting $k=1,\cdots,N$. We will refer to them by (2.7^h) and (2.8^h), respectively. Using (2.7) and (2.8) together with (2.7^h) and (2.8^h) we see immediately that

$$(2.12)\qquad \lambda_k \le \lambda_{h,k},\ k = 1,\cdots,N = \dim\ S_h.$$

We will assume that the family $\{S_h\}$ satisfies the approximability

assumption

(2.13) $$\varepsilon_u(h) = \|u\|_B^{-1} \inf_{\chi \in S_h} \|u-\chi\|_B \longrightarrow 0 \quad \text{as } h \longrightarrow 0, \text{ for each } u \in H_B.$$

It follows from (2.7), (2.8), (2.7^h), (2.8^h), and (2.13) that

(2.14) $$\lambda_{h,k} \longrightarrow \lambda_k \quad \text{as} \quad h \longrightarrow 0, \quad \text{for each} \quad k.$$

Finally we introduce

$$\bar{u}_j = \sqrt{\lambda_j}\, u_j ,$$

the exact eigenvectors normalized in $\|\cdot\|_D$, and

$$\bar{u}_{h,j} = \sqrt{\lambda_{h,j}}\, u_{h,j} ,$$

the approximate eigenvectors normalized in $\|\cdot\|_D$.

Throughout the paper, the specific eigenfunctions satisfying (2.6) ((2.11)) will be denoted by $u_j(u_{h,j})$. Thus the $u_j(u_{h,j})$ are normalized in $\|\cdot\|_B$; $\bar{u}_j(\bar{u}_{h,j})$ denotes the same eigenvectors, renormalized in $\|\cdot\|_D$. When we denote an eigenpair by (λ, u) we will not assume any particular normalization on u. C, C_i, d, and d_i will denote generic constants.

3. Preliminary Results

In this section we present several preliminary results that will be used in the sequel. For further information on eigenvalue problems we refer the reader to [4,8].

a) An Identity Relating the Eigenvalue and Eigenvector Errors

Here we present an identity that relates the errors in eigenvalue and eigenvector approximation.

Lemma 3.1. Suppose (λ, u) is an eigenpair of (2.5), suppose w is any vector in H_B with $\|w\|_D = 1$, and let $\lambda' = B(w,w)$. Then

$$\lambda' - \lambda = \|w-u\|_B^2 - \lambda\|w-u\|_D^2 . \tag{3.1}$$

Proof. By an easy calculation,

$$\|w-u\|_B^2 - \lambda\|w-u\|_D^2 = \|w\|_B^2 - 2B(w,u) + \|u\|_B^2 - \lambda\|w\|_D^2 + 2\lambda\, D(w,u) + \lambda\|u\|_D^2 . \tag{3.2}$$

Now

$$B(v,u) = \lambda D(v,u) \ \forall\ v \in H_B,$$

from which we get

$$B(w,u) = \lambda D(w,u) \tag{3.3}$$

and

$$\|u\|_B^2 = B(u,u) = \lambda D(u,u) = \lambda\|u\|_D^2 . \tag{3.4}$$

The result follows from (3.2)-(3.4) and the relations $\lambda' = \|w\|_B^2$ and $1 = \|w\|_D^2$. □

b) The Operators T and T_h

Let

$$H_D = \text{the completion of } H_B \text{ with respect } \|\cdot\|_D.$$

H_D is a Hilbert space with inner product D and, since $\|\cdot\|_D$ is assumed to be compact with respect to $\|\cdot\|_B$, H_B is compactly imbedded in H_D. (Alternatively, we could have assumed $H_B \subset H_D$, compactly, and let $D(u,v)$ be the inner product on H_D.)

From H_D and H_B construct the "negative space" $H_{-B} = H_B'$, with norm $\|\cdot\|_{-B}$. Then $H_D \subset H_{-B}$ compactly, and for $v \in H_B$, $D(u,v)$ has a continuous extension to $u \in H_{-B}$ so that $D(u,v)$ is continuous on $H_{-B} \times H_B$. For $u \in H_{-B}$, $\|u\|_{-B} = \sup_{v \in H_B} \dfrac{|D(u,v)|}{\|v\|_B}$. For a complete discussion of this construction we refer to [5, pp.31-39].

Next we introduce the operators $T, T_h \colon H_{-B} \longrightarrow H_B$ defined by

$$(3.5) \qquad \begin{cases} Tf \in H_B \\ B(Tf,v) = D(f,v), \ \forall\, v \in H_B, \end{cases}$$

$$(3.6) \qquad \begin{cases} T_h f \in S_h \\ B(T_h f,v) = D(f,v), \ \forall\, v \in S_h. \end{cases}$$

T and T_h are the solution and approximate solution operators for the "boundary value" problem corresponding to the eigenvalue problem (2.5). It follows immediately from (2.1), (2.2), and the fact that $D(f,v)$ is continuous on $H_{-B} \times H_B$ that T and T_h are bounded from H_{-B} to H_B. Since H_B is compactly imbedded in H_D and H_D is compactly imbedded in H_{-B}, T is compact from H_B to H_B, from H_D to H_D, and from H_{-B} to H_{-B}. T_h is, of course, also compact on H_B, H_D, and H_{-B}. It is easily seen that T and T_h are selfadjoint on H_D and that T is selfadjoint and positive definite on H_B (with respect to $B(u,v)$). It is immediate that T has eigenvalues

$$\mu_1 = \lambda_1^{-1} \geq \mu_2 = \lambda_2^{-1} \geq \cdots \searrow 0$$

and eigenvectors

$$u_1, u_2, \cdots,$$

and that T_h has eigenvalues

$$\mu_{h,1} = \lambda_{h,1}^{-1} \geq \cdots \geq \mu_{h,N} = \lambda_{h,N}^{-1}, \quad N = \dim S_h,$$

and eigenvectors

$$u_{h,1}, \cdots, u_{h,N}.$$

Let P_h be the orthogonal projection of H_B onto S_h; then from (3.6) we see that

$$T_h = P_h T.$$

Let

$$(3.7) \qquad \eta(h) = \|(I-P_h)T\|_{H_D \to H_B} = \|T-T_h\|_{H_D \to H_B} = \sup_{\substack{g \in H_D \\ \|g\|_D = 1}} \inf_{\chi \in S_h} \|Tg-\chi\|_B$$

and

$$(3.8) \qquad \nu(h) = \|(I-P_h)T\|_{H_B \to H_B} = \|T-T_h\|_{H_B \to H_B} = \sup_{\substack{g \in H_B \\ \|g\|_B = 1}} \inf_{\chi \in S_h} \|Tg-\chi\|_B .$$

Several of the results in Sections 4 and 5 are stated in terms of the qualities of η and ν. We now present some properties of η and ν.

<u>Lemma</u> 3.2. There are positive constants C_1 and C_2 such that

$$(3.9) \qquad C_1 \nu(h) \leq \eta(h) \leq C_2 \sqrt{\nu(h)} .$$

<u>Proof</u>. Since $\|u\|_D \leq C\|u\|_B \ \forall\ u \in H_B$ we have $\nu(h) \leq C\eta(h)$, which is the

first inequality in (3.9) with $C_1 = C^{-1}$. Now consider the second inequality in (3.9). From (3.5) and (3.6) we have

$$\|Tf\|_B \le \|f\|_{-B}, \quad \|T_h f\|_B \le \|f\|_{-B}$$

and hence

$$\|T-T_h\|_{H_{-B}\to H_B} \le 2 \tag{3.10}$$

and from (3.8) we have

$$\|T-T_h\|_{H_B\to H_B} = \nu(h). \tag{3.11}$$

We now note that H_{-B} and H_B are connected by a scale of Hilbert spaces. It thus follows from (3.10), (3.11), and a result on interpolation of linear operators [5, pp. 240-242] that

$$\eta(h) = \|T-T_h\|_{H_D\to H_B} \le C2^{1/2}\,\nu^{1/2} = C\nu(h)^{1/2},$$

which is the second inequality in (3.9). □

Lemma 3.3.

$$\lim_{h\to 0}\eta(h) = \lim_{h\to 0}\nu(h) = 0\ . \tag{3.12}$$

Proof: Because of Lemma 3.2 it is sufficient to show that $\lim_{h\to 0}\nu(h) = 0$. (2.13) implies that $P_h \longrightarrow I$ pointwise on H_B (in fact, (2.13) is equivalent to this result). Since $T: H_B \longrightarrow H_B$ is compact, $T\{g\in H_B: \|g\|_B = 1\}$ is relatively compact in H_B, and $\lim_{h\to 0}\nu(h) = 0$ follows from the standard result that a family of operators that converges pointwise on a space converges uniformly on a relatively compact subset. □

From Lemma 3.2 we have $\eta^2 \le O(\nu)$. It may happen that $\eta^2 = o(\nu)$. This is shown by the following example.

Example

Let

$$H_B = H_0^1(0,1),$$

$$B(u,v) = \int_0^1 a(x)\, u'v'dx,$$

and

$$D(u,v) = \int_0^1 u\, v\, dx,$$

where $0 < \alpha \le a(x) \le \beta < \infty$. ($H^\ell(0,1)$ is the ℓ^{th} order Sobolev space and $H_0^1(0,1) = \{u \in H^1(0,1)\colon u(0) = u(1) = 0\}$). For $f \in L_2(0,1)$, $u = Tf$ is the solution of

$$\begin{cases} -(a(x)u')' = f(x), \; 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases}$$

First suppose S_h = the space of C^0, piecewise linear functions with mesh size h that vanish at 0 and 1 and suppose $a(x)$ is smooth. Then we easily see that $\eta(h) \sim h$ and $\nu(h) \sim h$, so that $\eta^2 = o(\nu)$. Next suppose S_h = the space of C^0, piecewise quadratic functions vanishing at 0 and 1. If $a(x)$ is smooth we see that $\eta(h) \sim h$ and $\nu(h) \sim h^2$, so $\eta^2 \sim \nu$. If, on the other hand, $a(x)$ is rough, specifically if $a(x)$ is such that $g \in H_D = L_2(0,1)$ implies $u = Tf \in H^2(0,1)$, but $g \in H_B = H_0^1(0,1)$ does not, in general, imply $u \in H^\alpha(0,1)$ for $\alpha > 2$, then $\eta \sim h$ and $\nu \sim h$, so $\eta^2 = o(\nu)$.

From (2.13) we have

$$\|(I-P_h)u\|_B = \varepsilon_u(h)\ \|u\|_B \to 0, \quad \forall\ u \in H_B.$$

The usual dualiy argument (cf. Aubin [1], Nitsche [10], and Oganesjan-Rukhovets [11]) shows that $\|(I-P_h)u\|_D \le C\eta(h)\|(I-P_h)u\|_B$ and $\|(I-P_h)u\|_{-B} \le C\nu(h)\|(I-P_h)u\|_B$. For the sake of completeness we include proofs of these results.

Lemma 3.4.

(3.13a) $$\|(I-P_h)u\|_D \le \eta(h)\|(I-P_h)u\|_B, \ \forall\ u \in H_B$$

and

(3.13b) $$\|(I-P_h)u\|_{-B} \le \nu(h)\|(I-P_h)u\|_B, \ \forall\ u \in H_B.$$

Proof Since P_h is the orthogonal projection of H_B onto S_h, we have

$$B((I-P_h)u,Tg) = B((I-P_h)u,Tg-\chi), \quad \forall\ \chi \in S_h,$$

from which we get

(3.14) $$|B((I-P_h)u,Tg)| \le \|(I-P_h)u\|_B \inf_{\chi\in S_h} \|Tg-\chi\|_B .$$

From (3.5), the symmetry of D and B, and (3.14) we have

$$\|(I-P_h)u\|_D = \sup_{\substack{g\in H_D \\ \|g\|_D=1}} |D((I-P_h)u,g)|$$

$$= \sup_{\substack{g\in H_D \\ \|g\|_D=1}} |B((I-P_h)u,Tg)|$$

$$\le \sup_{\substack{g\in H_D \\ \|g\|_D=1}} \inf_{\chi\in S_h} \|Tg-\chi\|_B \, \|(I-P_h)u\|_B$$

$$\le \eta(h)\|(I-P_h)u\|_B,$$

which is (3.13a). Similarly,

$$\|(I-P_h)u\|_{-B} = \sup_{\substack{g\in H_B \\ \|g\|_B=1}} |D((I-P_h)u,g)|$$

$$= \sup_{\substack{g\in H_B \\ \|g\|_B=1}} \inf_{\chi\in S_h} \|Tg-\chi\|_B \, \|(I-P_h)u\|_B$$

$$= \nu(h)\|(I-P_h)u\|_B,$$

which is (3.13b). □

c) Preliminary Eigenvector Estimates

For $i = 1,2,\cdots$ let k_i be the lowest index of the i^{th} distinct eigenvalue of (2.5) and suppose λ_{k_i} has multiplicity q_i. Let $E = E(\lambda_{k_i})$ be the orthogonal projection of H_B onto $M(\lambda_{k_i})$ and let $E_h = E_h(\lambda_{k_i})$ be the orthogonal projection of H_B onto

(3.15) $\quad M_h = M_h(\lambda_{k_i})$ = the space of eigenvectors of (2.10) corresponding to $\lambda_{h,k_i+j}, j=0,\cdots,q_i-1$.

Lemma 3.5 There is a constant C_i such that

(3.16a) $$\|u-E_h(\lambda_{k_i})u\|_B \le C_i\|(I-P_h)u\|_B , \quad \forall\, u \in M(\lambda_{k_i}),$$

(3.16b) $$\|u-E_h(\lambda_{k_i})u\|_D \le C_i\|(I-P_h)u\|_D , \quad \forall\, u \in M(\lambda_{k_i}),$$

and

(3.16c) $$\|u-E_h(\lambda_{k_1})u\|_{-B} \le C_1\|(I-P_h)u\|_{-B}, \quad \forall\ u \in M(\lambda_{k_1}),$$

Proof. Suppose the spaces H_B, H_D, and H_{-B}, the bilinear forms B and D, and the operators T, T_h, E, and E_h have been complexified in the usual manner. Let Γ_{k_1} be a circle in the complex plane centered at $\mu_{k_1} = \lambda_{k_1}^{-1}$ enclosing no other eigenvalues of T. Then for h sufficiently small, $\mu_{h,k_1} = \lambda_{h,k_1}^{-1}, \cdots, \mu_{h,k_1+q_1-1} = \lambda_{h,k_1+q_1-1}^{-1}$, but no other eigenvalues of T_h, are contained in Γ_{k_1}, and

(3.17a) $$E(\lambda_{k_1}) = \frac{1}{2\pi i}\int_{\Gamma_{k_1}} (z-T)^{-1}\, dz$$

and

(3.17b) $$E_h(\lambda_{k_1}) = \frac{1}{2\pi i}\int_{\Gamma_{k_1}} (z-T_h)^{-1}\, dz.$$

These are the usual formulas for the spectral projections associated with T and μ_{k_1} and T_h and $\mu_{h,k_1}, \cdots, \mu_{h,k_1+q_1-1}$, respectively (cf. [9, Section X1.9]).

Consider now the proof of (3.16a). Using (3.17) we have

(3.18) $$\begin{aligned}\|u-E_h(\lambda_{k_1})u\|_B &= \|[E(\lambda_{k_1}) - E_h(\lambda_{k_1})]u\|_B \\ &= \left\|\frac{1}{2\pi i}\int_{\Gamma_{k_1}} [(z-T)^{-1} - (z-T_h)^{-1}]u\, dz\right\|_B \\ &= \frac{1}{2\pi}\left\|\int_{\Gamma_{k_1}} (z-T_h)^{-1}(T-T_h)(z-T)^{-1}u\, dz\right\|_B \\ &= \frac{1}{2\pi}\left\|\int_{\Gamma_{k_1}} (z-T_h)^{-1}(T-T_h)\frac{u}{z-\mu_{k_1}}\, dz\right\|_B\end{aligned}$$

$$\le \frac{1}{2\pi}\,[2\pi\ \mathrm{rad}\,(\Gamma_{k_1})] \sup_{\substack{z\in\Gamma_{k_1}\\ 0<h}} \|(z-T_h)^{-1}\|_{H_B\to H_B} \frac{\|(T-T_h)u\|_B}{\mathrm{rad}(\Gamma_{k_1})}$$

$$= \mu_{k_1} \sup_{\substack{z\in\Gamma_{k_1}\\ 0<h}} \|(z-T_h)^{-1}\|_{H_B\to H_B}\ \|(I-P_h)u\|_B,\ \forall\ u \in M(\lambda_{k_1}).$$

In the last inequality we used the relation $(T-T_h)u = (I-P_h)Tu = \mu_{k_1}(I-P_h)u$.

Now $\|T-T_h\|_{H_B\to H_B} \longrightarrow 0$ implies

$$C_1 = \mu_{k_1} \sup_{\substack{z\in\Gamma_{k_1}\\ 0<h}} \|(z-T_h)^{-1}\|_{H_B\to H_B} < \infty,$$

so we have established (3.16a).

Now consider the proof of (3.16b). The above analysis is relative to the space H_B (the integrals in (3.17) converge in the operator norm on H_B and $\|T-T_h\|_{H_B\to H_B} \longrightarrow 0.$). Since T and T_h can also be considered on H_D and $\|T-T_h\|_{H_D\to H_D} \longrightarrow 0$, we can apply the same argument in H_D. Note that the formulas (3.17) will now define projections on H_D which are extensions to H_D of E and E_h. We thus obtain (cf. (3.18))

$$\|u-E_h(\lambda_{k_1})u\|_D \le \mu_{k_1} \sup_{\substack{z\in\Gamma_{k_1}\\ h>0}} \|(z-T_h)^{-1}\|_{H_D\to H_D}\ \|(I-P_h)u\|_D,\ \forall\ u \in M(\lambda_{k_1}),$$

which is (3.16b).

The proof of (3.16c) is similar. □

<u>Remark 3.1</u>. It is essential in the Lemma 3.5 that h is sufficiently small, meaning small in comparison with the gap between λ_{k_1} and λ_{k_1-1}, λ_{k_1+1}. If this gap is small then it can happen that the approximation eigenfunction

u_{h,k_i} associated with λ_{h,k_i} could be close to u_{k_i-1} or u_{k_i+1}.

Lemma 3.5 is an eigenvector estimate since it provides an estimate for

u(an exact eigenvector) - E_hu(a linear combination of approximate eigenvectors).

We note that (2.13) and (3.16) imply that $E_h(\lambda_{k_i}): M(\lambda_{k_i}) \longrightarrow M_h(\lambda_{k_i})$ is one-to-one and onto for h sufficiently small.

We next prove a refinement of (3.16a) due to Chatelin [7,8]. (3.16a) shows that

$$\frac{\|u-E_h(\lambda_{k_i})u\|_B}{\|u-P_h u\|_B} \le O(1), \quad \forall\ u \in M(\lambda_{k_i}).$$

Chatelin showed that

$$\frac{\|u-E_h(\lambda_{k_i})u\|_B}{\|u-P_h u\|_B} \longrightarrow 1, \quad \text{as} \quad h \longrightarrow 0 (\text{see } (1.3a));$$

her argument, in fact, establishes

<u>Lemma 3.6</u> (Chatelin). There is a constant d_i such that

$$1 \le \frac{\|u-E_h(\lambda_{k_i})u\|_B}{\|u-P_h u\|_B} \le 1 + d_i \nu(h), \quad \forall\ u \in M(\lambda_{k_i}), \tag{3.19}$$

where $\nu(h)$ is defined in (3.8).

<u>Proof</u>. For the sake of completeness and to establish the form of the bound in the second inequality in (3.19) we present a proof of this result.

Let $\bar{T}_h = P_h\, T\, P_h = T_h\, P_h$. Note that $\bar{T}_h$ and T_h have the same nonzero eigenvalues, that $E_h(\lambda_{k_i})$ commutes with $\bar{T}_h$, and that $\bar{T}_h$ is selfadjoint with respect to B. For $u \in M(\lambda_{k_i})$,

$$(\bar{T}_h - \mu_{h,k_1})P_h u = P_h T(P_h - I)u + (\mu_{k_1} - \mu_{h,k_1})P_h u$$

and hence, since $E_h(\lambda_{k_1})$ commutes with $\bar{T}_h$,

$$(3.20) \qquad (\bar{T}_h - \mu_{h,k_1})(I-E_h(\lambda_{k_1}))P_h u = (I-E_h(\lambda_{k_1}))P_h T(P_h - I)u$$
$$+ (\mu_{k_1} - \mu_{h,k_1})(I-E_h(\lambda_{k_1}))P_h u.$$

Let Q be the orthogonal projection of H_B onto $\mathcal{N}(\bar{T}_h)$, the null space of $\bar{T}_h$. Then, any $z \in \mathcal{R}(I-E_h(\lambda_{k_1}))$, the range of $I-E_h(\lambda_{k_1})$, can be written as

$$z = \sum_{\substack{\ell=1 \\ \ell \neq k_1, \cdots, k_1+q_1-1}}^{N} B(z, u_{h,\ell})u_{h,\ell} + Qz.$$

Here we have used the orthogonal decomposition

$$H_B = \overline{\mathcal{R}(\bar{T}_h^*)} \oplus \mathcal{N}(\bar{T}_h)$$
$$= \overline{\mathcal{R}(\bar{T}_h)} \oplus \mathcal{N}(\bar{T}_h)$$
$$= \overline{\text{span}\,\{u_{h,1}, \cdots, u_{h,N}\}} \oplus \mathcal{N}(\bar{T}_h)$$
$$= \text{span}\,\{u_{h,1}, \cdots, u_{h,N}\} \oplus \mathcal{N}(\bar{T}_h).$$

Thus

$$(\bar{T}_h - \mu_{h,k_1})z = \sum_{\substack{\ell=1 \\ \ell \neq k_1, \cdots, k_1+q_1-1}}^{N} B(z, u_{h,\ell})(\mu_{h,\ell} - \mu_{h,k_1})u_{h,\ell} - \mu_{h,k_1}Qz,$$

and hence

$$(3.21) \qquad \|(\bar{T}_h - \mu_{h,k_1})z\|_B^2 = \sum_{\substack{\ell=1 \\ \ell \neq k_1, \cdots, k_1+q_1-1}}^{N} |B(z, u_{h,\ell})|^2 |\mu_{h,\ell} - \mu_{h,k_1}|^2$$

$$+ \ |\mu_{h,k_1}|^2 \ \|Qz\|_B^2$$

$$\geq \min \left\{ |\mu_{h,j} - u_{h,k_1}|^2, \ j=1,\cdots,N, j \neq k_i,\cdots,k_i+q_i-1, \ |\mu_{h,k_1}|^2 \right\}$$

$$\times \left\{ \sum_{\substack{\ell=1 \\ \ell \neq k_i,\cdots,k_i+q_i-1}}^{N} |B(z,u_{h,\ell})|^2 + \|Qz\|_B^2 \right\}$$

$$= \begin{cases} \min \left\{ |\mu_{h,k_1-1} - \mu_{h,k_1}|^2, |\mu_{h,k_1+1} - \mu_{h,k_1}|^2, |\mu_{h,k_1}|^2 \right\} \|z\|_B^2, & i \geq 2 \\ \min \left\{ |\mu_{h,k_2} - u_{h,k_1}|^2, \ |u_{h,k_1}|^2 \right\} \|z\|_B^2, & i = 1. \end{cases}$$

Since $\mu_{h,j} \to \mu_j$ (cf. (2.14)) for each j as $h \to 0$,

$$\begin{cases} \min \left\{ |\mu_{h,k_1-1} - \mu_{h,k_1}|^2, \ |\mu_{h,k_1+1} - \mu_{h,k_1}|^2, \ |\mu_{h,k_1}|^2 \right\}, & i \geq 2 \\ \min \left\{ |\mu_{h,k_2} - \mu_{h,k_1}|^2, \ |\mu_{h,k_1}|^2 \right\}, & i = 1 \end{cases}$$

$$\to \begin{cases} \min \left\{ |\mu_{k_1-1} - \mu_{k_1}|^2, \ |\mu_{k_1+1} - \mu_{k_1}|^2, \ |\mu_{k_1}|^2 \right\}, & i \geq 2 \\ \min \left\{ |\mu_{k_2} - \mu_{k_1}|^2, \ |\mu_{k_1}|^2 \right\}, & i = 1 \end{cases}$$

$$= \delta_i^2, \quad \text{as} \quad h \to 0,$$

from (3.21) we get

(3.22) $$\|(\bar{T}_h - \mu_{h,k_1})z\|_B \geq \delta_i \ \|z\|_B, \ \forall \ z \in \mathcal{R}(I - E_h(\lambda_{k_1})) \text{ and}$$
$$\forall \ \text{small} \ h,$$

where $\delta_i > 0$ and depends only on the gap between μ_{k_i} and μ_{k_i-1}, μ_{k_i+1}. Combining (3.20), (3.22), and the fact that $I-E_h(\lambda_{k_i})$ and P_h are orthogonal projections we have

$$\|(I-E_h(\lambda_{k_i}))\ P_h u\|_B \le \delta_i^{-1}\|(I-E_h(\lambda_{k_i}))\ P_h T(P_h-I)u$$

$$+\ (\mu_{k_i}-\mu_{h,k_i})(I-E_h(\lambda_{k_i}))\ P_h u\|_B$$

$$\le \delta_i^{-1}\Big\{\|T(P_h-I)^2 u\|_B +$$

$$|\mu_{k_i}-\mu_{h,k_i}|\ \|(I-E_h(\lambda_{k_i}))\ P_h u\|_B\Big\},$$

from which we get

$$\|(I-E_h(\lambda_{k_i}))\ P_h u\|_B \le d_i\|T(P_h-I)u\|_{H_B\to H_B}\ \|(P_h-I)u\|_B \tag{3.23}$$

$$= d_i\|(P_h-I)T\|_{H_B\to H_B}\|(P_h-I)u\|_B.$$

In the last equality we used the fact that (P_h-I) and T are selfadjoint and that the norm of an operator and its adjoint are equal.

(3.23) implies

$$\Big|\|(I-E_h(\lambda_{k_i})P_h)u\|_B - \|(I-P_h)u\|_B\Big| \le \|(I-E_h(\lambda_{k_i}))P_h u\|_B$$

$$\le d_i\|(P_h-I)T\|_{H_B\to H_B}\ \|(P_h-I)u\|_B\ ,$$

and hence

$$\left|\frac{\|(I-E_h(\lambda_{k_i})P_h)u\|_B}{\|(P_h-I)u\|_B} - 1\right| \le d_i\|(P_h-I)T\|_{H_B\to H_B}. \tag{3.24}$$

We easily see that

$$\|(I-P_h)u\|_B \le \|(I-E_h(\lambda_{k_i}))u\|_B \le \|(I-E_h(\lambda_{k_i})P_h)u\|_B ,$$

and thus

$$(3.25) \qquad 1 \le \frac{\|(I-E_h(\lambda_{k_i}))u\|_B}{\|(P_h-I)u\|_B} \le \frac{\|(I-E_h(\lambda_{k_i})P_h)u\|_B}{\|(I-P_h)u\|_B} .$$

Combining (3.24) and (3.25) we have

$$\begin{aligned} 0 &\le \frac{\|(I-E_h(\lambda_{k_i}))u\|_B}{\|(P_h-I)u\|_B} - 1 \\ &\le \frac{\|(I-E_h(\lambda_{k_i})P_h)u\|_B}{\|(P_h-I)u\|_B} - 1 \\ &\le d_i \|(P_h-I)T\|_{H_B \to H_B} , \quad \forall\, u \in M(\lambda_{k_i}). \end{aligned}$$

Recalling that $\|(P_h-I)T\|_{H_B \to H_B} = \nu(h)$ we obtain the desired result. □

Remarks 3.2. (3.19) should be compared with (4.21), which provides a stronger estimate for certain special u's in $M(\lambda_{k_i})$.

Lemmas 3.5 and 3.6 show that starting from any exact eigenvector u we can construct $E_h(\lambda_{k_i})u$, a linear combination of approximate eigenvectors that is close to u. One can also start with an approximate eigenvector and construct a close exact eigenvector. We present another result of Chatelin [7.8]; (see (1.3b)).

Lemma 3.7 (Chatelin). There is a constant d_i such

$$(3.26) \qquad 1 \le \frac{\|u_{h,j} - E(\lambda_{k_i})u_{h,j}\|_B}{\|P_h E(\lambda_{k_i})u_{h,j} - E(\lambda_{k_i})u_{h,j}\|_B} \le 1 + d_i \nu(h),$$

$$j = k_i, \cdots, k_i + q_i - 1.$$

Proof. Observing that $E(\lambda_{k_1})-E_h(\lambda_{k_1})P_h = (E_h(\lambda_{k_1})-E_h(\lambda_{k_1}))P_h + E(\lambda_{k_1})(I-P_h)$, we obtain

$$\|E(\lambda_{k_1})-E_h(\lambda_{k_1})P_h\|_{H_B\to H_B} \le \|E(\lambda_{k_1})-E_h(\lambda_{k_1})\|_{H_B\to H_B} + \|E(\lambda_{k_1})(I-P_h)\|_{H_B\to H_B}.$$

We easily see that

$$\begin{aligned}\|E(\lambda_{k_1})(I-P_h)\|_{H_B\to H_B} &= \|(I-P_h)E(\lambda_{k_1})\|_{H_B\to H_B}\\ &= \sup_{\substack{u\in H_B\\ \|u\|_B=1}} \|(I-P_h)E(\lambda_{k_1})u\|_B\\ &= \lambda_{k_1} \sup_{\substack{u\in H_B\\ \|u\|_B=1}} \|(I-P_h)TE(\lambda_{k_1})u\|_B\\ &\le \lambda_{k_1}\ \nu(h)\end{aligned}$$

and by a slight modification of estimate (3.18) we have

$$\|E(\lambda_{k_1})-E_h(\lambda_{k_1})\|_{H_B\to H_B} \le C\nu(h).$$

Thus

$$\|E(\lambda_{k_1})-E_h(\lambda_{k_1})P_h\|_{H_B\to H_B} \le C\nu(h). \tag{3.27}$$

Next note that

$$\begin{aligned}\left\{I-[E_h(\lambda_{k_1})P_h - E(\lambda_{k_1})]\right\} (u_{h,j} - E(\lambda_{k_1})u_{h,j})\\ = [E_h(\lambda_{k_1})P_h-I]E(\lambda_{k_1})u_{h,j}.\end{aligned}$$

Hence, using (3.24) and (3.27), we have

$$\|u_{h,j} - E(\lambda_{k_1})u_{h,j}\|_B \le \left\|\left\{I-[E_h(\lambda_{k_1})P_h-E(\lambda_{k_1})]\right\}^{-1}\right\|_{H_B\to H_B}$$

$$\times \; \| [E_h(\lambda_{k_i})P_h - I]E(\lambda_{k_i})u_{h,j}\|_B$$

$$\le \frac{\| [E_h(\lambda_{k_i})P_h - I]E(\lambda_{k_i})u_{h,j}\|_B}{1 - \|E_h(\lambda_{k_i})P_h - E(\lambda_{k_i})\|_{H_B \to H_B}}$$

$$\le \frac{(1+d_i\nu)\|(P_h - I)E(\lambda_{k_i})u_{h,j}\|_B}{1 - C\nu},$$

which implies the second inequality in (3.26). The first is immediate. □

d) Relation between Eigenvector Error in $\|\cdot\|_B$, $\|\cdot\|_D$, and $\|\cdot\|_{-B}$.

In subsection 3.b) we noted that $\|(I-P_h)u\|_D \le \eta(h)\ \|(I-P_h)u\|_B$ and $\|(I-P_h)u\|_{-B} \le \nu(h)\|(I-P_h)u\|_B$. In this subsection we establish similar results for the eigenvector error.

For $i = 1,2,\cdots$ and $j = k_i,\cdots,k_i + q_i - 1$, let $\bar{u}_j^h \in M(\lambda_{k_i})$ satisfy $E_h(\lambda_{k_i})\ \bar{u}_j^h = \bar{u}_{h,j}$. We know from the discussion in Subsection 3.c) that $\bar{u}_j^h$ exists and is unique for h small. From (3.13a) and (3.16b) we have

$$\|\bar{u}_j^h - \bar{u}_{h,j}\|_D = \|\bar{u}_j^h - E_h(\lambda_{k_i})\bar{u}_j^h\|_D$$

$$\le C_i\ \eta(h)\|(I-P_h)\bar{u}_j^h\|_B$$

$$\le C_i\ \eta(h)\|\bar{u}_j^h - \bar{u}_{h,j}\|_B,$$

or

(3.28a)
$$\frac{\|\bar{u}_j^h - \bar{u}_{h,j}\|_D}{\|\bar{u}_j^h - \bar{u}_{h,j}\|_B} \le C_i\eta(h).$$

It follows immediately (by scaling) that

$$(3.28b) \qquad \frac{\|u_j^h - u_{h,j}\|_D}{\|u_j^h - u_{h,j}\|_B} \le C_1 \eta(h) \ ,$$

where $u_j^h \in M(\lambda_{k_1})$ satisfies $E_h(\lambda_{k_1})u_j^h = u_{h,j}$. (Recall that $\|\bar{u}_{j,h}\|_D = 1$ and $\|u_{j,h}\|_B = 1$.) Similarly, from (3.13b) and (3.16c) we get

$$(3.29a) \qquad \frac{\|\bar{u}_j^h - \bar{u}_{h,j}\|_{-B}}{\|\bar{u}_j^h - \bar{u}_{h,j}\|_B} \le C_1 \nu(h)$$

and

$$(3.29b) \qquad \frac{\|u_j^h - u_{h,j}\|_{-B}}{\|u_j^h - u_{h,j}\|_B} \le C_1 \nu(h).$$

By Lemma 3.1 we know that

$$(3.30) \qquad \lambda_{h,j} - \lambda_{k_1} = \|u-\bar{u}_{h,j}\|_B^2 - \lambda_{k_1}\|u-\bar{u}_{h,j}\|_D^2$$

$$= \|u-\bar{u}_{h,j}\|_B^2 \left\{1-\lambda_{k_1} \frac{\|u - \bar{u}_{h,j}\|_D^2}{\|u - \bar{u}_{h,j}\|_B^2}\right\}, \ \forall \ u \in M(\lambda_{k_1}).$$

As u varies over $M(\lambda_{k_1})$ it is clear from (3.30) that $\dfrac{\|u - \bar{u}_{h,j}\|_D^2}{\|u - \bar{u}_{h,j}\|_B^2}$ is minimized for that $\bar{u}_0$ that minimizes $\|u-\bar{u}_{h,j}\|_B^2$, namely for $u_0 = E(\lambda_{k_1})\ \bar{u}_{h,j}$. Thus we have

$$(3.31a) \qquad \frac{\|E(\lambda_{k_1})\bar{u}_{h,j}-\bar{u}_{h,j}\|_D}{\|E(\lambda_{k_1})\bar{u}_{h,j}-\bar{u}_{h,j}\|_B} \le \frac{\|\bar{u}_j^h - \bar{u}_{h,j}\|_D}{\|\bar{u}_j^h - \bar{u}_{h,j}\|_B} \le C_1 \eta(h).$$

We, of course, also get

(3.31b)
$$\frac{\|E(\lambda_{k_i})u_{h,j}-u_{h,j}\|_D}{\|E(\lambda_{k_i})u_{h,j}-u_{h,j}\|_B} \le C_i\eta(h).$$

Estimates (3.31) are similar to (3.28), but involve a different pairing of approximate and exact eigenvectors.

Remarks 3.4. Pierce and Varga [12] proved eigenvector estimates in $\|\cdot\|_D$ and Babuška and Osborn [6] established them in $\|\cdot\|_{-B}$.

4. Precise Asymptotic Estimates for the Eigenvalue and Eigenvector Error

In this section we use the notation introduced in Subsection 3.c), i.e., we let k_i be the lowest index of the i^{th} distinct eigenvalue of (2.5) and suppose λ_{k_i} has multiplicity q_i.

a) The Eigenvalue Error

For $i=1,2,\cdots$ and $j=k_i,\cdots,k_i+q_i-1$ fixed, Chatelin [7,8] has shown that

$$(4.1)\qquad \frac{(\lambda_{h,j}-\lambda_{k_i})/\lambda_{k_i}}{\|(I-P_h)E(\lambda_{k_i})u_{h,j}\|_B^2 \,/\, \|E(\lambda_{k_i})u_{h,j}\|_B^2} \longrightarrow 1, \quad \text{as} \quad h \longrightarrow 0 \ (\text{cf. } 1.3c)).$$

We now prove a refinement of (4.1) (cf.(1.3c) and (1.4))

<u>Theorem 4.1</u>. For $i = 1,2,\cdots$ there is a constant d_i such that

$$(4.2)\qquad \left|\frac{(\lambda_{h,j}-\lambda_{k_i})/\lambda_{k_i}}{\|(I-P_h)E(\lambda_{k_i})u_{h,j}\|_B^2 \,/\, \|E(\lambda_{k_i})u_{h,j}\|_B^2} - 1\right| \le d_i\eta^2(h), \quad j=k_i,\cdots,k_i+q_i-1,$$

where $\eta(h)$ is defined in (3.7).

<u>Proof</u> Let $u = E(\lambda_{k_i})u_{h,j}$. We have

$$\begin{aligned}(4.3)\qquad (\mu_{k_i}-\mu_{h,j})B(u,u_{h,j}) &= B(Tu,u_{h,j}) - B(u,T_hu_{h,j})\\ &= B(u,(T-T_h)u_{h,j})\\ &= B(T(I-P_h)u,u_{h,j})\\ &= B(T(I-P_h)u,u) + B(T(I-P_h)u,u_{h,j}-u)\\ &= B(T(I-P_h)^2u,u) + B(T(I-P_h)u,u_{h,j}-u)\end{aligned}$$

$$= \mu_{k_i} B((I-P_h)u,(I-P_h)u) + D((I-P_h)u, u_{h,j}-u).$$

Using the fact that $B(u,u_{h,j}) = B(u,E(\lambda_{k_i})u_{h,j}) = \|u\|_B^2$, (4.3) can be written as

$$\frac{\lambda_{h,j}-\lambda_{k_i}}{\lambda_{h,j}\lambda_{k_i}} \|u\|_B^2 = \frac{1}{\lambda_{k_i}} \|(I-P_h)u\|_B^2 + D((I-P_h)u, u_{h,j}-u).$$

Dividing by $\|(I-P_h)u\|_B^2$, multiplying by $\lambda_{h,j}$, and subtracting 1 from both sides we find

$$\text{(4.4)} \qquad \frac{(\lambda_{h,j}-\lambda_{k_i})/\lambda_{k_i}}{\|(I-P_h)u\|_B^2 \,/\, \|u\|_B^2} - 1 = \frac{\lambda_{h,j}-\lambda_{k_i}}{\lambda_{k_i}} + \lambda_{h,j} \frac{D((I-P_h)u, u_{h,j}-u)}{\|(I-P_h)u\|_B^2}.$$

From (4.1) or the standard, well known results for eigenvalue approximation we have

$$\text{(4.5)} \qquad \frac{\lambda_{h,j}-\lambda_{k_i}}{\lambda_{k_i}} \le d_i \left[\sup_{\substack{u\in M(\lambda_{k_i}) \\ \|u\|_B=1}} \|(I-P_h)u\|_B \right]^2$$

$$= d_i \left[\sqrt{\lambda_{k_i}} \sup_{\substack{u\in M(\lambda_{k_i}) \\ \|u\|_D=1}} \|(I-P_h)Tu\|_B \right]^2$$

$$\le d_i \eta^2(h), \quad j = k_i, \cdots, k_i + q_i - 1,$$

from (3.13a) we have

$$\text{(4.6)} \qquad \|(I-P_h)u\|_D \le \eta(h) \, \|(I-P_h)u\|_B,$$

and from (3.26) and (3.31b) we have

$$\text{(4.7)} \qquad \|u_{h,j}-u\|_D = \|u_{h,j}-E(\lambda_{k_i})u_{h,j}\|_D$$

$$\le d_1 \ \eta(h) \|u_{h,j} - E(\lambda_{k_1}) u_{h,j}\|_B$$

$$= d_1 \ \eta(h) \|(I-P_h)u\|_B \ .$$

Combining (4.4) - (4.7) we obtain

$$\left| \frac{(\lambda_{h,j}-\lambda_{k_1})/\lambda_{k_1}}{\|(I-P_h)u\|_B^2 \ / \ \|u\|_B^2} - 1 \right| \le d_1 \eta^2(h) + \frac{\lambda_{h,j} |D((I-P_h)u, u_{h,j}-u|}{\|(I-P_h)u\|_B^2}$$

$$\le d_1 \eta^2 + \frac{\lambda_{h,j} \|(I-P_h)u\|_D \|u_{h,j}-u\|_D}{\|(I-P_h)u\|_B^2}$$

$$\le d_1 \eta^2,$$

the desired result. □

Remark 4.1. Formula (4.4) is due to Chatelin [5,6], and is used by her to prove (4.1). Using eigenvector estimates in $\|\cdot\|_B$ ((3.26)) one can prove

$$\left| \frac{(\lambda_{h,j}-\lambda_{k_1})/\lambda_{k_1}}{\|(I-P_h)E(u_{k_1})u_{h,j}\|_B^2 \ / \ \|E(\lambda_{k_1})u_{h,j}\|_B^2} - 1 \right| \le d_1 \nu(h).$$

(4.2), which was proved using eigenvector estimates in $\|\cdot\|_D$ ((3.31b) together with (3.26)), is an improvement over this result since, as we saw in Subsection 3.b), η^2 may be of higher order than ν.

Theorem 4.1 relates the eigenvalue error $(\lambda_{h,j}-\lambda_{k_1})/\lambda_{k_1}$ to $\|(I-P_h)u\|_B^2$, with $u = E(\lambda_{k_1})u_{h,j}$. We now prove a result that relates the eigenvalue error to $\|(I-P_h)u\|_B^2 \ / \ \|u\|_B^2$, where $u \in M(\lambda_{k_1})$ and $E_h(\lambda_{k_1}) \ u = u_{h,j}$, i.e., $u = u_j^h$, as defined in Subsection 3.d).

Theorem 4.2. For $i = 1,2,\cdots$ there is a constant d_i such that

$$(4.8)\qquad \left| \frac{(\lambda_{h,j}-\lambda_{k_i})/\lambda_{k_i}}{\|(I-P_h)u\|_B^2 \,/\, \|u\|_B^2} - 1 \right| \le d_i\ \eta^2(h),\ j = k_i,\cdots,\ k_i + q_i-1,$$

where $u \in M(\lambda_{k_i})$ satisfies $E_h(\lambda_{k_i})u = u_{h,j}$.

Proof. With $u \in M(\lambda_{k_i})$ satisfying $E_h(\lambda_{k_i})u = u_{h,j}$ we have

$$\begin{aligned}(\mu_{k_i}-\mu_{h,j})\ B\ (u,u_{h,j}) &= B(Tu,u_{h,j}) - B(u,T_h u_{h,j}) \\ &= B(T(I-P_h)u,u) \\ &\qquad + B(T(I-P_h)u,u_{h,j}-u) \\ &= \mu_{k_i}\|(I-P_h)u\|_B^2 \\ &\qquad + B(T(I-P_h)u,u_{h,j}-u),\end{aligned}$$

from which we get, as above,

$$(4.9)\qquad \frac{(\lambda_{h,j}-\lambda_{k_i})/\lambda_{k_i}}{\|(I-P_h)u\|_B^2 \ /\ \|E_h(\lambda_{k_i})u\|_B^2} - 1 = \frac{\lambda_{h,j}-\lambda_{k_i}}{\lambda_{k_i}} + \lambda_{h,j}\ \frac{D((I-P_h)u,u_{h,j}-u)}{\|(I-P_h)u\|_B^2}.$$

It follows from (3.13a) and (3.16b) that

$$(4.10)\qquad \|u_{h,j} - u\|_D \le d_i\eta\|(I-P_h)u\|_B\ .$$

Combining (4.5), (4.6), (4.9), and (4.10) we obtain

$$\left| \frac{(\lambda_{h,j}-\lambda_{k_i})/\lambda_{k_i}}{\|(I-P_h)u\|_B^2 \ /\ \|E_h(\lambda_{k_i})u\|_B^2} - 1 \right| \le d_i\ \eta^2(h),$$

from which we get

$$(4.11)\qquad \left|\frac{(\lambda_{h,j}-\lambda_{k_i})/\lambda_{k_i}}{\|(I-P_h)u\|_B^2\ /\ \|u\|_B^2} - \frac{\|u\|_B^2}{\|E_h(\lambda_{k_i})u\|_B^2}\right| \le d_i\eta^2(h)\frac{\|u\|_B^2}{\|E_h(\lambda_{k_i})u\|_B^2}\ .$$

Since $u = (u-E_h(\lambda_{k_i})u) + E_h(\lambda_{k_i})u$ is an orthogonal decomposition in H_B, we have

$$\|u\|_B^2 = \|u-E_h(\lambda_{k_i})u\|_B^2 + \|E_h(\lambda_{k_i})u\|_B^2$$

and hence

$$(4.12)\qquad \frac{\|u\|_B^2}{\|E_h(\lambda_{k_i})u\|_B^2} = 1 + \frac{\|u-E_h(\lambda_{k_i})u\|_B^2}{\|E_h(\lambda_{k_i})u\|_B^2}\ .$$

Using (3.16a) and (2.13) we see that

$$(4.13)\qquad \frac{\|u-E_h(\lambda_{k_i})u\|_B^2}{\|E_h(\lambda_{k_i})u\|_B^2} \le C\ \varepsilon_u^2(h)$$

$$\le C\ \eta^2(h).$$

Combining (4.11), (4.12), and (4.13) we get the desired result. □

b) The Eigenvector Error

Let $i = 1,2,\cdots$ and let $j = k_i,\cdots,k_i + q_i - 1$ be fixed and consider $\bar{u}_{h,j}$ and $E(\lambda_{k_i})\bar{u}_{h,j}$ (recall that $\|\bar{u}_{h,j}\|_D = 1$). We showed in Subsection 3.d) (see (3.31a)) that

$$(4.14)\qquad \|E(\lambda_{k_i})\bar{u}_{h,j}-\bar{u}_{h,j}\|_D \le d_i\eta(h)\|E(\lambda_{k_i})\bar{u}_{h,j}-\bar{u}_{h,j}\|_B\ .$$

From Lemma 3.1 we have

$$(4.15)\qquad \lambda_{h,j} - \lambda_{k_i} = \|E(\lambda_{k_i})\bar{u}_{h,j}-\bar{u}_{h,j}\|_B^2 - \lambda_{k_i}\|E(\lambda_{k_i})\bar{u}_{h,j}-\bar{u}_{h,j}\|_D^2\ .$$

Combining (4.14) and (4.15) we obtain

$$\lambda_{h,j} - \lambda_{k_i} \ge \|E(\lambda_{k_i})\bar{u}_{h,j}-\bar{u}_{h,j}\|_B^2\ (1-d_i\eta^2(h)),$$

which implies

$$(4.16) \qquad \frac{\|E(\lambda_{k_i})\bar{u}_{h,j}-\bar{u}_{h,j}\|_B^2}{\|(I-P_h)E(\lambda_{k_i})\bar{u}_{h,j}\|_B^2} \le \frac{\lambda_{h,j}-\lambda_{k_i}}{\|(I-P_h)E(\lambda_{k_i})\bar{u}_{h,j}\|_B^2\,(1-d_i\eta^2)} .$$

Since $\bar{u}_{h,j} = E(\lambda_{k_i})\bar{u}_{h,j} + (\bar{u}_{h,j}-E(\lambda_{k_i})\bar{u}_{h,j})$ is an orthogonal decomposition in H_D, we have

$$1 = \|E(\lambda_{k_i})\bar{u}_{h,j}\|_D^2 + \|\bar{u}_{h,j} - E(\lambda_{k_i})\bar{u}_{h,j}\|_D^2 .$$

From this, (3.26), and (4.14) we get

$$(4.17) \quad 1 \le \|E(\lambda_{k_i})\bar{u}_{h,j}\|_D^2 + d_i\eta^2\|(P_h-I)E(\lambda_{k_i})\bar{u}_{h,j}\|_B^2 = \frac{\|E(\lambda_{k_i})\bar{u}_{h,j}\|_B^2}{\lambda_{k_i}}\,(1+d_i\eta^4).$$

Now, combining (4.2), (4.16), and (4.17) we have

$$\begin{aligned}
\frac{\|E(\lambda_{k_i})\bar{u}_{h,j}-\bar{u}_{h,j}\|_B}{\|(I-P_h)E(\lambda_{k_i})\bar{u}_{h,j}\|_B} &\le \left\{\frac{(\lambda_{h,j}-\lambda_{k_i})/\lambda_{k_i}}{\|(I-P_h)E(\lambda_{k_i})\bar{u}_{h,j}\|_B^2\,/\|E(\lambda_{k_i})\bar{u}_{h,j}\|_B^2} \times \frac{1+d_i\eta^4}{1-d_i\eta^2}\right\}^{1/2} \\
&\le \left\{(1+d_i\eta^2)\,\frac{(1+d_i\eta^4)}{1-d_i\eta^2}\right\}^{1/2} \\
&\le [1+d_i\eta^2]^{1/2} \\
&\le 1 + d_i\eta^2(h).
\end{aligned}$$

We summarize this (cf. (1.3b) and (1.4)) in

Theorem 4.3. For $i = 1,2,\cdots$ there is a constant d_i such that

$$(4.18) \qquad 1 \le \frac{\|E(\lambda_{k_i})\bar{u}_{h,j}-\bar{u}_{h,j}\|_B}{\|(I-P_h)E(\lambda_{k_i})\bar{u}_{h,j}\|_B} \le 1 + d_i\eta^2(h), \quad j = k_i,\cdots,k_i+q_i-1.$$

$\bar{u}_{h,j}$ can be replaced by $u_{h,j}$ in (4.18).

Remark 4.2 (4.18) is stronger than (3.26) since η^2 may be of higher order than ν.

Next consider $\bar{u}_{h,j}$ and $\bar{u}_j^h$ (recall that $\bar{u}_j^h \in M(\lambda_{k_i})$ satisfies $E_h(\lambda_{k_i})\bar{u}_j^h = \bar{u}_{h,j}$). We know (see (3.28a)) that

$$\|\bar{u}_j^h - \bar{u}_{h,j}\|_D \le d_i\eta(h)\ \|\bar{u}_j^h - \bar{u}_{h,j}\|_B.$$

This, together with Lemma 3.1, yields

$$\begin{aligned}\lambda_{h,j} - \lambda_{k_i} &= \|\bar{u}_j^h - \bar{u}_{h,j}\|_B^2 - \lambda_{k_i}\|\bar{u}_j^h - \bar{u}_{h,j}\|_D^2 \\ &\ge \|\bar{u}_j^h - \bar{u}_{h,j}\|_B^2\,(1-d_i\eta^2),\end{aligned}$$

which implies

$$(4.19)\qquad \begin{aligned}\frac{\|\bar{u}_j^h-\bar{u}_{h,j}\|_B^2}{\|(I-P_h)\bar{u}_j^h\|_B^2} &\le \frac{\lambda_{h,j}-\lambda_{k_i}}{\|(I-P_h)\bar{u}_j^h\|_B^2\,(1-d_i\eta^2)} \\ &\le \frac{(\lambda_{h,j}-\lambda_{k_i})/\lambda_{k_i}}{\dfrac{\|(I-P_h)\bar{u}_j^h\|_B^2}{\|\bar{u}_j^h\|_B^2}\,(1-d_i^2\eta\ )}\ .\end{aligned}$$

Finally, combining (4.8) and (4.19) we have

$$\begin{aligned}1 \le \frac{\|\bar{u}_j^h-\bar{u}_{h,j}\|_B}{\|(I-P_h)\bar{u}_j^h\|_B} &\le \left(\frac{1+d_i\eta^2}{1-d_i\eta^2}\right)^{1/2} \\ &\le 1 + d_i\eta^2.\end{aligned}$$

This result (cf. (1.3a) and (1.4)) and the related result (3.19) are summarized in

Theorem 4.4. For $i = 1,2,\cdots$ there is a constent d_i such that

$$(4.20) \qquad 1 \le \frac{\|\bar{u}_j^h - \bar{u}_{h,j}\|_B}{\|(I-P_h)\bar{u}_j^h\|_B} = \frac{\|\bar{u}_j^h - E_h(\lambda_{k_i})\bar{u}_j^h\|_B}{\|(I-P_h)\bar{u}_j\|_B} \le 1 + d_i\eta^2(h),$$

$$j = k_i,\cdots,k_i+q_i-1.$$

(4.20) remains valid if $\bar{u}_{h,j}$ is replaced by $u_{h,j}$ and $\bar{u}_j^h$ by u_j^h. There is a constant d_i such that

$$(4.21) \qquad 1 \le \frac{\|u-E_h(\lambda_{k_i})u\|_B}{\|(u-P_h)u\|_B} \le 1 + d_i\nu(h), \text{ for all } u \in M(\lambda_{k_i}).$$

Remark 4.3. We have restated (3.19) in (4.21) because it is related to (4.20) and it is the strongest known result of its specific type. It should be noted that (4.21) is true for all $u \in M(\lambda_{k_i})$, whereas (4.20) is valid only for $u = \bar{u}_j^h$, $j = k_i,\cdots,k_i + q_i-1$. However, for these u's, (4.20) is stronger than (4.21).

Remark 4.4 See [2,4] for a numerical study of the reliability of the results of this section - which are of an asymptotic nature - as a guide to practical computations - which often takes place in the preasymptotic phase.

5. An Additional Result for Multiple Eigenvalues

Theorem 4.1 shows that

$$\lambda_{h,k_i} - \lambda_{k_i} \le C \frac{\inf_{\chi\in S_h} \|E(\lambda_{k_i})u_{h,k_i} - \chi\|_B^2}{\|E(\lambda_{k_i})u_{h,k}\|_B^2} .$$

In [3] Babuška and Osborn proved the stronger result (cf. (1.5))

$$\lambda_{h,k_i} - \lambda_{k_i} \le C \inf_{\substack{u\in M(\lambda_{k_i}) \\ \|u\|_B=1}} \inf_{\chi\in S_h} \|u-\chi\|_B^2$$

(as well as similar estimates for $\lambda_{h,j} - \lambda_{k_i}$, $j = k_i + 1,\cdots,k_i + q_i-1$, and for the eigenvector errors), which shows that $\lambda_{h,k_i} - \lambda_{k_i}$, the error in the approximate eigenvalue closest to λ_{k_i}, is governed by the approximability of the exact eigenvector corresponding to λ_{k_i} that can be best approximated by S_h. In this section we give a simplified proof of the results of [3], which in addition provides information on C (the results in [3] only established that C is a constant), and estimate the eigenvector error in $\|\cdot\|_D$ and $\|\cdot\|_{-B}$.

As above, for $i = 1,2,\cdots$ suppose k_i is the lowest index of the i^{th} distinct eigenvalue of (2.5) and let q_i be its multiplicity, i.e., suppose

$$\lambda_{k_{i-1}+q_{i-1}-1} = \lambda_{k_i-1} < \lambda_{k_i} = \lambda_{k_i+1} =\cdots=\lambda_{k_i+q_i-1} < \lambda_{k_i+q_i} = \lambda_{k_{i+1}}.$$

Let

$$(5.1) \qquad \begin{aligned} \varepsilon_{i,j}(h) &= \inf_{\substack{u\in M(\lambda_{k_i}) \\ \|u\|_B=1 \\ B(u,u_{h,k_i})=\cdots=B(u,u_{h,k_i+j-2}) = 0}} \inf_{\chi\in S_h} \|u-\chi\|_B \\ &= \inf_{\substack{u\in M(\lambda_{k_i}) \\ \|u\|_B=1 \\ B(u,u_{h,k_i})=\cdots=B(u,u_{h,k_i+j-2}) = 0}} \varepsilon_u(h), \qquad j=1,\cdots,q_i, \end{aligned}$$

where $M(\lambda_{k_i})$ is defined in (2.9). The restrictions $B(u,u_{h,k_i}) =\cdots= B(u,u_{h,k_i+j-2}) = 0$ are considered vacous if $j = 1$. We note that they are equivalent to

$$B(u,E(\lambda_{k_i})u_{h,\ell}) = 0, \ \ell=k_i,\cdots,k_i + j - 2$$

and to

$$B(E_h(\lambda_{k_i})u, u_{h,\ell}) = 0, \quad \ell = k_i, \cdots, k_i + j - 2.$$

Theorem 5.1 (cf. (1.5)) For $i = 1, 2, \cdots$ there is a function $C_i(h)$ and a constant $\bar{C}_i$, with

$$(5.2) \qquad \bar{C}_i(h) \le 1 + d_i \nu(h), \qquad d_i = \text{constant},$$

such that

$$(5.3) \qquad (\lambda_{h,k_i+j-1} - \lambda_{k_i})/\lambda_{k_i} \le \bar{C}_i(h)\, \varepsilon^2_{i,j}(h), \quad j=1, \cdots, q_i,$$

and such that the eigenvectors $u_1, u_2, \cdots$ of (2.5) can be chosen so that (2.6) is satisfied and such that

$$(5.4) \qquad \|u_{h,k_i+j-1} - u_{k_i+j-1}\|_B \le \bar{C}_i(h)\, \varepsilon_{i,j}(h), \quad j=1, \cdots, q_i,$$

$$(5.5a) \qquad \|u_{h,k_i+j-1} - u_{k_i+j-1}\|_D \le C_i \eta(h) \varepsilon_{i,j}(h), \quad j=1, \cdots, q_i,$$

and

$$(5.5b) \qquad \|u_{h,k_i+j-1} - u_{k_i+j-1}\|_{-B} \le C_i \nu(h) \varepsilon_{i,j}(h), \quad j=1, \cdots, q_i,$$

where $\nu(h)$ and $\eta(h)$ are defined in (3.7) and (3.8).

Proof. Let i and j, with $i = 1, 2, \cdots$ and $j = 1, \cdots, q_i$ be fixed. Note that $\varepsilon_u(h) \le \lambda_{k_i} \nu(h)$, $\forall\, u \in M(\lambda_{k_i})$ and $\varepsilon_{i,j}(h) \le \lambda_{k_i} \nu(h)$, $j = 1, \cdots, q_i$. Let $u \in M(\lambda_{k_i})$ with $B(u, u_{h,k_i}) = \cdots = B(u, u_{h,k_i+j-2}) = 0$ and $\|u\|_B = 1$. Now apply (2.7^h) and Lemma 3.1 with $(\lambda, u) = \left(\lambda_{k_i}, \dfrac{u}{\|E_h(\lambda_{k_i})u\|_D}\right)$ and $w = \dfrac{E_h(\lambda_{k_i})u}{\|E_h(\lambda_{k_i})u\|_D}$. Since

$$B(E_h(\lambda_{k_i})u, u_{h,\ell}) = 0, \quad \ell = 1, \cdots, k_i - 1$$

by the orthogonality of the approximate eigenvectors, and

$$B(E_h(\lambda_{k_i})u, u_{h,\ell}) = B(u, u_{h,\ell}) = 0, \quad \ell = k_i, \cdots, k_i+j-2$$

by the assumption on u, we have

$$(5.6) \qquad \lambda_{h,k_i+j-1} - \lambda_{k_i} \le B\left[\frac{E_h(\lambda_{k_i})u}{\|E_h(\lambda_{k_i})u\|_D}, \frac{E_h(\lambda_{k_i})u}{\|E_h(\lambda_{k_i})u\|_D}\right] - \lambda_{k_i}$$

$$= \frac{\|E_h(\lambda_{k_i})u-u\|_B^2 - \lambda_{k_i}\|E_h(\lambda_{k_i})u-u\|_D^2}{\|E_h(\lambda_{k_i})u\|_D^2}$$

$$\le \frac{\|E_h(\lambda_{k_i})u-u\|_B^2}{\|E_h(\lambda_{k_i})u\|_D^2} .$$

From (3.19) we have

$$(5.7) \qquad \|E_h(\lambda_{k_i})u-u\|_B \le (1+d\nu)\|u-P_hu\|_B .$$

From (3.13) and (3.16b) we see that

$$\left|\, \|E_h(\lambda_{k_i})u\|_D - \lambda_{k_i}^{-1/2} \right| = \left|\|E_h(\lambda_{k_i})u\|_D - \|u\|_D\right|$$

$$\le \|E_h(\lambda_{k_i})u-u\|_D$$

$$\le d\ \|u - P_hu\|_D$$

$$\le d\eta\ \|u-P_hu\|_B$$

$$= d\eta(h)\ \varepsilon_u(h),$$

which shows that

$$(5.8) \qquad \frac{1}{\lambda_{k_i}\|E_h(\lambda_{k_i})u\|_D^2} \le \left[1 + \frac{d\eta\varepsilon_u(h)}{\|E_h(\lambda_{k_i})u\|_D}\right]^2$$

$$\le 1 + d\eta\ \varepsilon_u(h)$$

$$\le 1 + d\eta\, \nu.$$

Combining (5.6) - (5.8) we get

$$(5.9) \qquad (\lambda_{h,k_i+j-1} - \lambda_{k_i})/\lambda_{k_i} \le (1 + d\nu)^2(1+d\eta\nu)\ \|u-P_h u\|_B^2$$

$$\le (1 + d\nu)\ \|u-P_h u\|_B^2 .$$

Now since (5.9) holds for all $u \in M(\lambda_{k_i})$ with $B(u,u_{h,\ell}) = 0$, $\ell = k_i,\cdots,k_i+j-2$, and $\|u\|_B = 1$, we have

$$(\lambda_{h,k_i+j-1} - \lambda_{k_i})/\lambda_{k_i} \le (1+d\nu(h)) \left[\inf_{\substack{u\in M(\lambda_{k_i}) \\ \|u\|_B=1 \\ B(u,u_{h,k_i})=\cdots=B(u,u_{h,k_i+j-2})=0}} \ \inf_{\chi\in S_h} \|u-\chi\|_B \right]^2$$

$$= (1+d_i\nu(h))\varepsilon_{i,j}^2(h),$$

which is (5.3) with $C_i(h) = 1+d_i\nu(h)$. Thus (5.2) and (5.3) have both been proved.

<u>Remark 5.1.</u> (2.7^h) and Lemma 3.1 lead to a particularly simple proof of a result slightly weaker than (5.3) for the case $i = j = 1$. It follows immediately from these two results that

$$\lambda_{h,1} - \lambda_1 \le B\left(\frac{P_h u}{\|P_h u\|_D}, \frac{P_h u}{\|P_h u\|_D}\right) - \lambda_1$$

$$= \frac{\|P_h u-u\|_B^2 - \lambda_1 \|P_h u-u\|_D^2}{\|P_h u\|_D^2}$$

$$\le \frac{\|P_h u-u\|_B^2}{\|P_h u\|_D^2}\ \forall\ u \in M(\lambda_1),$$

and hence

$$(\lambda_{h,1}-\lambda_1)/\lambda_1 \le \inf_{\substack{u\in M(\lambda_1)\\ \|u\|_B=1}} \frac{\|P_h u-u\|_B^2}{\lambda_1\|P_h u\|_D^2} \le C(h)\ \varepsilon_{1,1}(h),$$

where $C(h) \longrightarrow 1$.

Now consider (5.4) and (5.5). Let $i = 1,2,\cdots$ and $j=1,\cdots,q_i$ be fixed. Let $u'_{h,k_i+j-1} \in M(\lambda_{k_i})$ satisfy $E_h(\lambda_{k_i})u'_{k_i+j-1} = u_{h,k_i+j-1}$ ($u'_{k_i+j-1} = u^h_{k_i+j-1}$, where $u^h_{k_i+j-1}$ was introduced in Subsection 3.d)). Applying Lemma 3.1 with

$$(\lambda,u) = \left(\lambda_{k_i},\ \frac{u'_{k_i+j-1}}{\|u_{h,k_i+j-1}\|_D}\right) \quad\text{and}\quad w = \frac{u_{h,k_i+j-1}}{\|u_{h,k_i+j-1}\|_D}$$

we get

$$\lambda_{h,k_i+j-1} - \lambda_{k_i} = \left\| \frac{u_{h,k_i+j-1}}{\|u_{h,k_i+j-1}\|_D} - \frac{u'_{k_i+j-1}}{\|u_{h,k_i+j-1}\|_D} \right\|_B^2 - \lambda_{k_i}\left\| \frac{u_{h,k_i+j-1}}{\|u_{h,k_i+j-1}\|_D} - \frac{u'_{k_i+j-1}}{\|u_{h,k_i+j-1}\|_D} \right\|_D^2 . \tag{5.10}$$

From (3.28b) we have

$$\|u'_{k_i+j-1} - u_{h,k_i+j-1}\|_D \le C\eta(h)\|u'_{k_i+j-1} - u_{h,k_i+j-1}\|_B . \tag{5.11}$$

(5.10) and (5.11) yield

$$\lambda_{h,k_i+j-1} - \lambda_{k_i+j-1} \ge \frac{[1-\lambda_{k_i}C^2\eta^2(h)]}{\|u_{h,k_i+j-1}\|_D^2}\|u_{h,k_i+j-1} - u'_{k_i+j-1}\|_B^2,$$

which, together with (2.11), (2.12), and (5.3), yields

$$\text{(5.12)} \qquad \|u'_{k_i+j-1} - u_{h,k_i+j-1}\|_B \le \frac{(\lambda_{h,k_i+j-1} - \lambda_{k_i})^{1/2}}{\lambda_{h,k_i+j-1}^{1/2}\,[1-\lambda_{k_i}C^2\eta^2]^{1/2}}$$

$$\le \frac{\lambda_{k_i}^{1/2}C_i(h)^{1/2}\varepsilon_{i,j}(h)}{\lambda_{h,k_i+j-1}^{1/2}\,[1-\lambda_{k_i}C^2\eta^2]^{1/2}}$$

$$\le \bar{C}_i(h)\ \varepsilon_{i,j}(h),\ \ j = 1,\cdots,q_i,$$

where, because of (5.2), $\bar{C}_i(h) \le 1 + d_i\nu(h)$. (5.12) shows that the u'_{k_i+j-1} satisfy estimates (5.4). (5.12), together with (3.28b) and (3.29b), shows that the u'_{k_i+j-1} satisfy estimates (5.5). They will not in general, however, be orthonormal with respect to B, so that (2.6) may not be satisfied.

It remains to modify the u'_{k_i+j-1}, i.e., replace u'_{k_i+j-1} by u_{k_i+j-1}, in such a way that (2.6) and (5.4) and (5.5) hold. We proceed by induction on j. Let $j = 1$. If we define $u_{k_i} = \dfrac{u'_{k_i}}{\|u'_{k_i}\|_B}$, we have $\|u_{k_i}\|_B = 1$, so that (2.6) is satisfied. From (5.12) we have

$$\text{(5.13)} \qquad \left|\|u'_{k_i+j-1}\|_B - 1\right| = \left|\left[1 + \|u'_{k_i+j-1} - u_{h,k_i+j-1}\|_B^2\right]^{1/2} - 1\right|$$

$$\le \frac{\|u'_{k_i+j-1} - u_{h,k_i+j-1}\|_B^2}{2}$$

$$\le C\nu\ \varepsilon_{i,j},\ \ j=1,\cdots,q_i,$$

and hence

$$\|u_{k_i} - u_{h,k_i}\|_B \le \|u_{k_i} - u'_{k_i}\|_B + \|u'_{k_i} - u_{h,k_i}\|_B$$

$$\leq \left| \|u'_{k_i}\|_B - 1 \right| + \|u'_{k_i} - u_{h,k_i}\|_B$$

$$\leq C\nu\varepsilon_{i,1} + \bar{C}_i(h)\eta\varepsilon_{i,1}$$

$$\leq \bar{C}_i(h)\varepsilon_{i,1}(h),$$

where $\bar{C}_i(h) \leq 1 + d_i\nu(h)$, which is (5.4) for $j = 1$. Using (3.9), (5.13), and the fact that the u'_{k_i+j-1} satisfy (5.5a) we get

$$\|u_{k_i} - u_{h,k_i}\|_D \leq \|u_{k_i} - u'_{k_i}\|_D + \|u'_{k_i} - u_{h,k_i}\|_D$$

$$= \lambda_{k_i}^{-1} \left| \|u'_{k_i}\|_B - 1 \right| + \|u'_{k_i} - u_{h,k_i}\|_D$$

$$\leq C\nu\varepsilon_{i,1} + C\eta\varepsilon_{i,1}$$

$$\leq C_i\eta(h)\varepsilon_{i,1}(h),$$

which is (5.5a) for $j = 1$. A similar estimate establishes (5.56) for $j = 1$.

Next suppose $j = 2$. Let $u''_{k_i} = u'_{k_i+1} - B(u'_{k_i+1}, u_{k_i})u_{k_i}$. Using (5.4) for $j = 1$, (5.5b) for $j = 1$, (5.12), and the facts that (5.5b) holds for the u'_{k_i+j-1} and that $\varepsilon_{i,1} \leq \varepsilon_{i,2}$, we have

(5.14) $$|B(u'_{k_i+1}, u_{k_i})| \leq |B(u'_{k_i+1} - u_{h,k_i+1}, u_{k_i})| + |B(u_{h,k_i+1} - u'_{k_i+1}, u_{k_i} - u_{h,k_i})|$$

$$+ |B(u'_{k_i+1}, u_{k_i} - u_{h,k_i}|$$

$$= \lambda_{k_i}|D(u'_{k_i+1} - u_{h,k_i+1}, u_{k_i})|$$

$$+ |B(u_{h,k_i+1} - u'_{k_i+1}, u_{k_i} - u_{h,k_i})|$$

$$+ \lambda_{k_i+1}|D(u'_{k_i+1}, u_{k_i} - u_{h,k_i})|$$

$$\le \lambda_{k_i} \|u'_{k_i+1} - u_{h,k_i+1}\|_{-B} \|u_{k_i}\|_B$$

$$+ \|u_{h,k_i+1} - u'_{k_i+1}\|_B \|u_{k_i} - u_{h,k_i}\|_B$$

$$+ \lambda_{k_i+1} \|u'_{k_i+1}\|_B \|u_{k_i} - u_{h,k_i}\|_{-B}$$

$$\le \lambda_{k_i} \|u'_{k_i+1} - u_{h,k_i+1}\|_{-B}$$

$$+ \|u_{h,k_i+1} - u'_{k_i+1}\|_B \|u_{k_i} - u_{h,k_i}\|_B$$

$$+ \lambda_{k_i} \left(1 + \|u'_{k_i+1} - u_{h,k_i+1}\|_B\right) \|u_{k_i} - u_{h,k_i}\|_{-B}$$

$$\le C\nu\varepsilon_{i,2} + \bar{C}_i(h)\varepsilon_{i,2}\, \bar{C}_i(h)\varepsilon_{i,1} + C\nu\bar{C}_i(h)\varepsilon_{i,1}$$

$$\le C\nu(h)\varepsilon_{i,2}(h),$$

and hence

(5.15) $$\|u'_{k_i+1} - u''_{k_i+1}\|_B = |B(u'_{k_i+1}, u_{k_i})| \le C\nu(h)\varepsilon_{i,2}(h).$$

Now set $u_{k_i+1} = \dfrac{u''_{k_i+1}}{\|u''_{k_i+1}\|_B}$. Combining (5.12), (5.13), and (5.15) we obtain

$$\|u_{k_i+1} - u_{h,k_i+1}\|_B \le \|u_{k_i+1} - u''_{k_i+1}\|_B + \|u''_{k_i+1} - u_{h,k_i+1}\|_B$$

$$= \left|\|u''_{k_i+1}\|_B - 1\right| + \|u''_{k_i+1} - u_{h,k_i+1}\|_B$$

$$\le \left|\|u'_{k_i+1}\|_B - 1\right| + 2\|u''_{k_i+1} - u'_{k_i+1}\|_B$$

$$+ \|u'_{k_i+1} - u_{h,k_i+1}\|_B$$

$$+ C\nu\varepsilon_{i,2}(h) + C\nu\,\varepsilon_{i,2}(h) + \bar{C}_i(h)\varepsilon_{i,2}(h)$$

$$\le \bar{C}_i(h)\varepsilon_{i,2}(h),$$

where $\bar{C}_i(h) \le 1 + d_i\nu(h)$, which is (5.4) for $j = 2$.

Now consider (5.5a) for $j = 2$. Using (5.13), (5.14), (5.15) and the fact that the u'_{k_i+j-1} satisfy (5.5) we have

$$\begin{aligned}\|u_{k_i+1}-u_{h,k_i+1}\|_D &\le \|u_{k_i+1}-u''_{k_i+1}\|_D + \|u''_{k_i+1}-u_{h,k_i+1}\|_D \\ &= \lambda_{k_i}^{-1/2}\ \big|\ \|u''_{k_i+1}\|_B-1\big| + \|u''_{k_i+1}-u_{h,k_i+1}\|_D \\ &\le \lambda_{k_i}^{-1/2}\ \big|\ \|u'_{k_i+1}\|_B-1\big| + \lambda_{k_i}^{-1/2}\|u''_{k_i+1}-u'_{k_i+1}\|_B \\ &\quad + \|u'_{k_i+1} - u_{h,k_i+1}\|_D + |B(u'_{k_i+1},u_{k_i})|\,\|u_{k_i}\|_D \\ &\quad + C\nu\varepsilon_{i,2} + C\nu\varepsilon_{i,2} + C\eta\varepsilon_{i,2} + C\nu\varepsilon_{i,2} \\ &\le C_i\eta\varepsilon_{i,2}(h),\end{aligned}$$

which is (5.5a) for $j = 2$. The proof of (5.5b) is similar.

Continuing in this manner we get (2.6), (5.4), and (5.5) for $j=1,\cdots,q_i$. This completes the proof. □

<u>Theorem 5.2</u>. For $i = 1,2,\cdots$ there is a function $\hat{C}_i(h)$ with

(5.16) $$\hat{C}_i(h) \ge 1 - d_i\nu(h), \quad d_i \ge 0, \text{ constant},$$

such that

(5.17) $$(\lambda_{h,k_i+j-1} - \lambda_{k_i})/\lambda_{k_i} \ge \hat{C}_i(h)\ \varepsilon^2_{i,j}(h), \quad j = 1,\cdots,q_i,$$

and

(5.18) $$\|u_{h,k_i+j-1} - u_{k_i+j-1}\|_B \ge \hat{C}_i(h)\ \varepsilon_{i,j}(h), \quad j = 1,\cdots,q_i.$$

<u>Proof</u>. First consider (5.18) for $j = 1$. It is immediate that

$$\|u_{h,k_1} - u_{k_1}\|_B \geq \inf_{\substack{u\in M(\lambda_{k_1})\\ \|u\|_B=1}} \inf_{\chi\in S_h} \|u-\chi\|_B$$

$$= \varepsilon_{i,1}(h).$$

Thus for $j = 1$, (5.18) holds with $\hat{C}_i(h) = 1$.

Now suppose $j = 2$. Since

$$\begin{aligned} B(u'_{k_1+1}, u_{h,k_1}) &= B(u'_{k_1+1}, E_h(\lambda_{k_1})u_{h,k_1}) \\ &= B(E_h(\lambda_{k_1})u'_{k_1+1}, u_{h,k_1}) \\ &= B(u_{h,k_1+1}, u_{h,k_1}) \\ &= 0, \end{aligned}$$

we see that

$$\|u_{h,k_1+1}-u'_{k_1+1}\|_B \geq \inf_{\substack{u\in M(\lambda_{k_1})\\ \|u\|_B=1\\ B(u,u_{h,k_1})=0}} \inf_{\chi\in S_h} \|u-\chi\|_B = \varepsilon_{i,2}(h).$$

Combining this result with (5.13) and (5.15) we get

$$\begin{aligned} \|u_{h,k_1+1}-u_{k_1+1}\|_B &\geq \|u_{h,k_1+1}-u'_{k_1+1}\|_B - \|u'_{k_1+1}-u''_{k_1+1}\|_B - \|u''_{k_1+1}-u_{k_1+1}\|_B \\ &\geq \|u_{h,k_1+1}-u'_{k_1+1}\|_B - 2\|u'_{k_1+1}-u''_{k_1+1}\|_B - \big|\|u'_{k_1+1}\|_B-1\big| \\ &\geq (1-d_i\nu)\varepsilon_{i,2}(h), \end{aligned}$$

which is (5.18) for $j = 2$. Continuing in this manner we get (5.18) for $j=1,\cdots,q_i$.

Now consider (5.17). From Lemma 3.1, (5.5a), and (5.18) we see that

$$(\lambda_{h,k_1+j-1} - \lambda_{k_1})/\lambda_{k_1} = \frac{\|u_{h,k_1+j-1} - u_{k_1+j-1}\|_B^2}{\lambda_{k_1}\|u_{h,k_1+j-1}\|_D^2}$$

$$- \frac{\|u_{h,k_i+j-1} - u_{k_i+j-1}\|_D^2}{\|u_{h,k_i+j-1}\|_D^2}$$

$$\geq \frac{\lambda_{h,k_i+j-1}}{\lambda_{k_i}} \left[(1-d_i\nu)^2 - \lambda_{k_i}\bar{C}_i^2\eta^2\right]\varepsilon_{i,j}^2 ,$$

which implies (5.17).

Remark 5.2. Note that in Theorems 5.1 and 5.2 we have shown that

$$\left|\frac{(\lambda_{h,k_i+j-1} - \lambda_{k_i})/\lambda_{k_i}}{\varepsilon_{i,j}^2(h)} - 1\right| \leq d_i\nu(h),$$

whereas in Theorems 4.1 and 4.2 we showed that

$$\left|\frac{(\lambda_{h,j} - \lambda_{k_i})/\lambda_{k_i}}{\|(I-P_h)E(\lambda_{k_i})u_{h,j}\|_B^2/\|E(\lambda_{k_i})u_{h,j}\|_B^2} - 1\right| \leq d_i\eta^2(h),$$

and

$$\left|\frac{(\lambda_{h,j} - \lambda_{k_i})/\lambda_{k_i}}{\|(I-P_h)u\|_B^2/\|u\|_B^2} - 1\right| \leq d_i\eta^2(h),$$

for $u \in M(\lambda_{k_i})$ with $E_h(\lambda_{k_i})u = u_{h,j}$.

Remarks 5.3. For a computational illustration of the results in this section see [3,4].

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Further information may be obtained from Professor I. Babuška, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.